Certificates of positivity and polynomial minimization in the multivariate Bernstein basis

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Motivation

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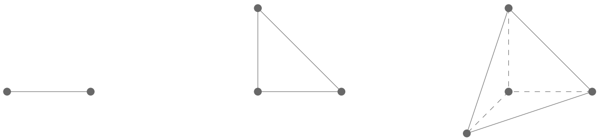
- non degenerate simplex $V = \mathrm{Conv}\,[V_0, \ldots, V_k] \subset \mathbb{R}^k$
- barycentric coordinates λ_i $(i = 0, \ldots, k)$:
  - polynomials of degree 1
  - $\sum \lambda_i = 1$
  - $x \in V \Leftrightarrow \forall i,\ \lambda_i(x) \geq 0$

Motivation

Example : standard simplex

$$\Delta = \{x \in \mathbb{R}^k \mid \forall i, x_i \geq 0 \text{ et } \sum x_i = 1\}$$



$x \geq 0$
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  $\hookrightarrow$ epidemiology problems

Outline

1 Multivariate Bernstein basis

2 Standard triangulation

3 Control polytope : approximation and convergence

4 Certificates of positivity

5 Polynomial minimization

Bernstein polynomials

Notations

- multi-index $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{N}^{k+1}$
- $|\alpha| = \alpha_0 + \cdots + \alpha_k = d$
- multinomial coefficient $\binom{d}{\alpha} = \dfrac{d!}{\alpha_0! \ldots \alpha_k!}$

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Bernstein polynomials of degree d with respect to V

$$B_\alpha^d = \binom{d}{\alpha} \lambda^\alpha = \binom{d}{\alpha} {\lambda_0}^{\alpha_0} \ldots {\lambda_k}^{\alpha_k}.$$

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Appear naturally in the expansion

$$1 = 1^d = (\lambda_0 + \cdots + \lambda_k)^d = \sum_{|\alpha|=d} \binom{d}{\alpha} \lambda^\alpha = \sum_{|\alpha|=d} B_\alpha^d.$$

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$\hookrightarrow$ Bernstein coefficients :

$$f = \sum_{|\alpha|=d} b_\alpha(f, d, V) B_\alpha^d.$$

- $b(f, d, V)$: list of coefficients $b_\alpha = b_\alpha(f, d, V)$

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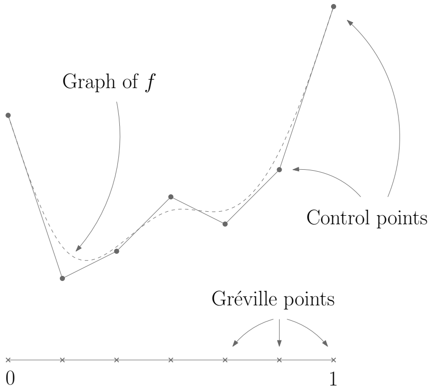

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$\hookrightarrow$ bound on the gap between $f\left(N_\alpha\right)$ and b_α

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- Result : These simplices form a triangulation of $\mathcal{C}$.

Kuhn's triangulation of the cube

In dimension 2 and 3

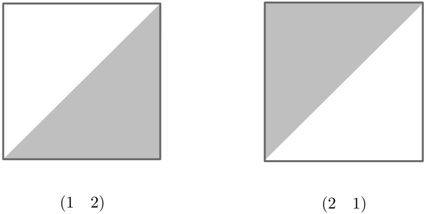

FIGURE: Kuhn's triangulation in dimension 2

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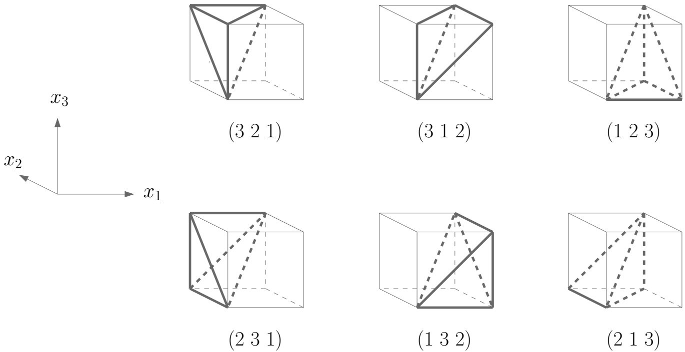

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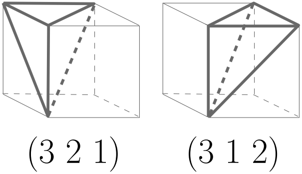

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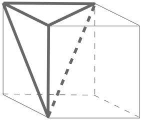

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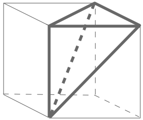

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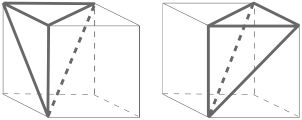
$$\Downarrow$$
$$V_{p-1}, V_p^\sigma, V_p^{\sigma'}, V_{p+1} \text{ form a parallelogram.}$$



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  - Then define the simplex $V^F = V_0^F + \frac{1}{d} V^{\sigma_F} =$
    $$\left[V_0^F, V_0^F + \frac{e_{\sigma_F(1)}}{d}, \ldots, V_0^F + \frac{e_{\sigma_F(1)}}{d} + \cdots + \frac{e_{\sigma_F(k)}}{d}\right].$$

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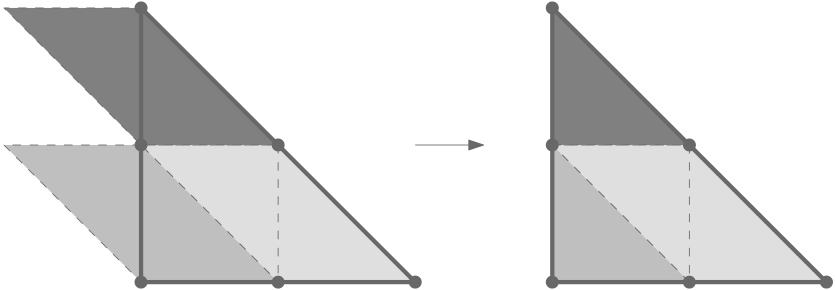
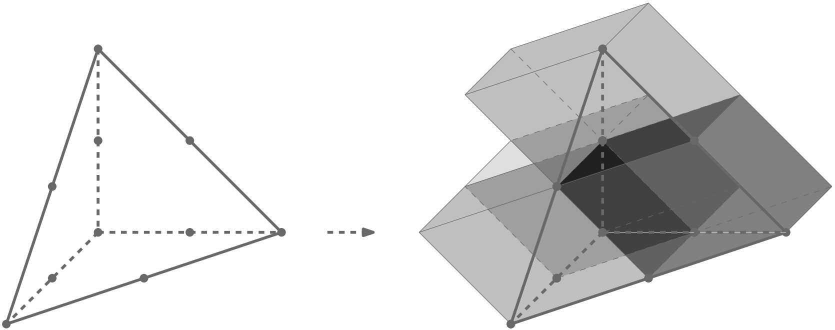
- The standard triangulation $T_d(V)$ of degree d of any simplex $V \subset \mathbb{R}^k$ is then obtained by affine transformation.

Standard triangulation of a simplex

In dimension 2, degree 2



Standard triangulation of a simplex
In dimension 3, degree 2



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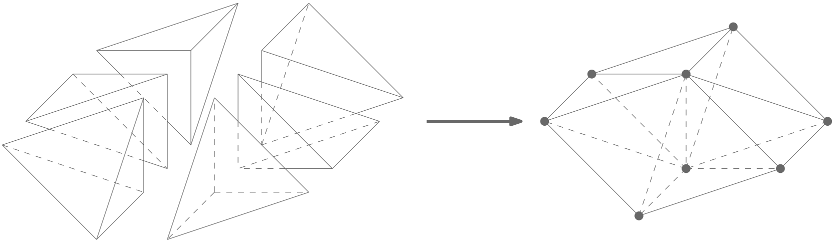

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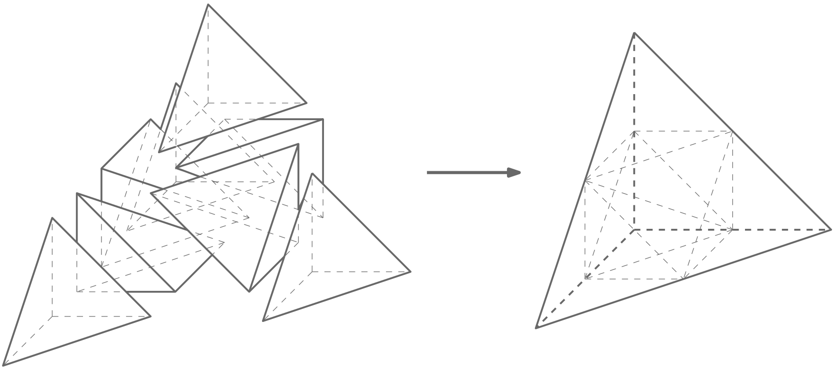

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Standard triangulation of a simplex
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- Here, we will consider standard triangulations of degree 2^N as consecutive standard triangulations of degree 2.

Control polytope

Definition

Let f be a polynomial of degree d, and V a simplex.
The control polytope associated to f and V is the unique continuous function $\hat{f}$, piecewise linear over each simplex of the standard triangulation of degree d of V and satisfaying the following interpolation property :

$$\forall |\alpha| = d, \ \hat{f}(N_\alpha) = b_\alpha.$$

Control polytope

Convexity property

Control polytope
Convexity property

Theorem

The following are equivalent :

(i) The control polytope $\hat{f}$ is convex.

(ii) $\forall|\gamma| = d-2,\ \forall\ 0 \leq i < j \leq k,$

$$\underbrace{b_{\gamma+e_i+e_{j-1}} + b_{\gamma+e_{i-1}+e_j} - b_{\gamma+e_i+e_j} - b_{\gamma+e_{i-1}+e_{j-1}}}_{\Delta_2 b_{\gamma,i,j}(f,d,V)} \geq 0.$$

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- $\Delta_2 b_{\gamma,i,j}(f, d, V)$: second differences

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- $\Delta_2 b_{\gamma,i,j}(f,d,V)$: second differences
- $\|\Delta_2 b(f,d,V)\|_\infty = \max_{\substack{|\gamma|=d-2 \\ 0\leq i<j\leq k}} |\Delta_2 b_{\gamma,i,j}(f,d,V)|$.

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Approximation

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Theorem ('08)

$$\max_{|\alpha|=d} |f(N_\alpha) - b_\alpha| \leq \frac{k(k+2)}{24} d \|\Delta_2 b(f, d, V)\|_\infty.$$

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$$\max_{|\alpha|=d} |f(N_\alpha) - b_\alpha| \leq \frac{k(k+2)}{24} d \|\Delta_2 b(f, d, V)\|_\infty.$$

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- Reif ('00) : bound $(d/3) \|\Delta_2 b\|_\infty$ when $k = 2$.

Control polytope

Convergence under degree elevation

Control polytope

Convergence under degree elevation

Theorem ('08)

Let $f \in \mathbb{R}[X]$ of degree d, over the standard simplex Δ.
Let $N \geq 1$. Then :

$$\max_{|\alpha|=2^N d} \left| f\left(\frac{\alpha_1}{2^N d}, \ldots, \frac{\alpha_k}{2^N d}\right) - b_\alpha(f, 2^N d, \Delta) \right| \leq \frac{k(k+2)}{24} \frac{d(d-1)}{2^N d - 1} \|\Delta_2 b(f, d, \Delta)\|_\infty.$$

Control polytope
Convergence under subdivision

Control polytope

Convergence under subdivision

Theorem ('08)

Let $f \in \mathbb{R}[X]$ of degree d, over the standard simplex Δ.
Let $U \in T_{2N}(\Delta)$ $(N \geq 1)$. Then :

$$\max_{|\alpha|=d} |f(N_\alpha) - b_\alpha(f, d, U)|$$

$$\leq \left(\frac{k(k+2)}{24}\right)^2 \frac{dk(k+1)(k+3)}{2^{2N}} \|\Delta_2 b(f, d, \Delta)\|_\infty .$$

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Certificate of positivity in the Bernstein basis

$$Cert(f, d, \Delta) : \begin{cases} \forall |\alpha| = d, \quad b_\alpha \geq 0 \\ \forall i = 0, \dots, k, \quad b_{de_i} > 0 \end{cases} \Rightarrow f > 0 \text{ on } \Delta$$

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- Warning : The converse is false in general !
  - $f = 6x^2 - 6x + 2 > 0$ on $[0, 1]$, but $b(f, 2, [0, 1]) = [2, -1, 2]$.

4 Certificates of positivity

- By degree elevation
- By subdivision

Certificates of positivity
by degree elevation

Idea : Express f in the Bernstein basis of increasing degree.

Theorem ('08)

$$2^N > \frac{k(k+2)}{24}(d-1)\frac{\|\Delta_2 b(f,d,\Delta)\|_\infty}{m} \Rightarrow Cert(f, 2^N d, \Delta).$$

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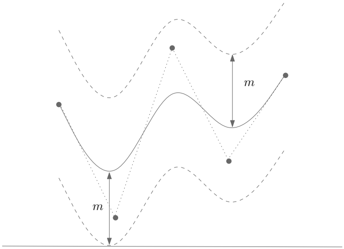

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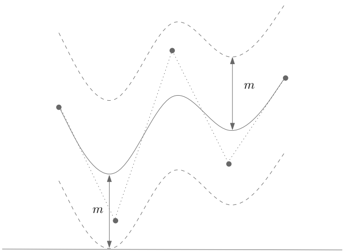

Powers, Reznick ('03)

$$2^N > \frac{(d-1)}{2}\frac{\max_{|\alpha|=d} |b_\alpha(f,d,\Delta)|}{m}$$

$$\Rightarrow Cert(f, 2^N d, \Delta).$$

4 Certificates of positivity

- By degree elevation
- By subdivision

Certificates of positivity
by subdivision

Idea : Keep the degree constant, and subdivide Δ.

Tool : successive standard triangulations of degree 2.

Theorem ('08)

$$2^N > \frac{k(k+2)}{24}\sqrt{dk(k+1)(k+3)}\sqrt{\frac{\|\Delta_2 b(f,d,\Delta)\|_\infty}{m}}$$

$\Rightarrow \forall U \in T_{2^N}(\Delta),\ Cert(f,d,U)$ holds.

Certificates of positivity
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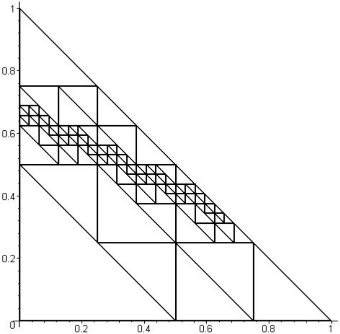

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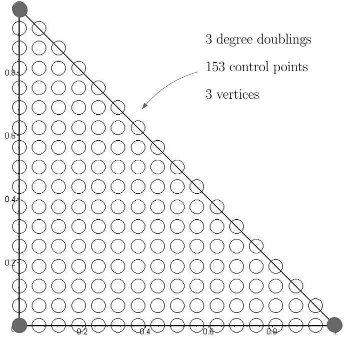

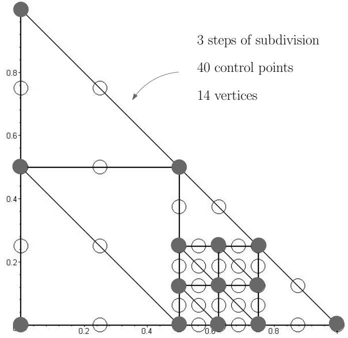

Certificates of positivity

by subdivision

The process stops :

Theorem

There exists an explicit $m_{k,d,\tau} > 0$ such that if f has degree $\leq d$ and the bitsize of its coefficients is bounded by τ, then

$$f > m_{k,d,\tau} \text{ on } \Delta.$$

Enclosing property

Let m denote the minimum of f over the standard simplex Δ.

Goal : Enclose m with an arbitrary precision.

Enclosing property

If V is a simplex, and m_V the minimum of f over V, then :

$$m_V \in [s_V, t_V],$$

where
$$\begin{cases} s_V = \min b_\alpha = b_\beta \text{ for some } \beta \\ t_V = \min[f(N_\beta), \underbrace{b_{de_i}}_{=f(V_i)}, i = 0, \ldots, k] \end{cases}$$

Algorithm

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- Subdivide : $\Delta = V^1 \cup \cdots \cup V^s$.

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$$t_{V^i} - s_{V^i} < \varepsilon,$$

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Tool : Successive standard triangulations of degree 2.

Algorithm

The process stops :

Theorem ('08)

If $2^N > \frac{k(k+2)}{24}\sqrt{dk(k+1)(k+3)}\sqrt{\frac{\|\Delta_2 b(f,d,\Delta)\|_\infty}{\varepsilon}}$,

then at most N steps of subdivision are needed.

Conclusions & future work

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