Certificates of positivity and polynomial minimization in the multivariate Bernstein basis

SAGA Kick Off - November '08

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Motivation

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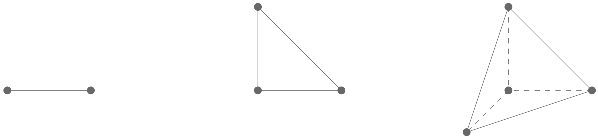
- non degenerate simplex $V = \text{Conv}\,[V_0, \ldots, V_k] \subset \mathbb{R}^k$
- barycentric coordinates λ_i $(i = 0, \ldots, k)$:
  - polynomials of degree 1
  - $\sum \lambda_i = 1$
  - $x \in V \Leftrightarrow \forall i,\ \lambda_i(x) \geq 0$

Motivation

Example : standard simplex

$$\Delta = \{x \in \mathbb{R}^k \mid \forall i, x_i \geq 0 \text{ et } \sum x_i = 1\}$$



$$x \geq 0 \qquad 1 - x \geq 0$$

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- Decide if f is positive on V (or not)
- Obtain a simple proof
  $\hookrightarrow$ certificate of positivity
- Compute the minimum of f over V (and localize the minimizers)

Outline

Bernstein polynomials

Notations

- multi-index $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{N}^{k+1}$
- $|\alpha| = \alpha_0 + \cdots + \alpha_k = d$
- multinomial coefficient $\binom{d}{\alpha} = \dfrac{d!}{\alpha_0! \ldots \alpha_k!}$

Bernstein polynomials

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Bernstein polynomials of degree d with respect to V

$$B_\alpha^d = \binom{d}{\alpha} \lambda^\alpha = \binom{d}{\alpha} {\lambda_0}^{\alpha_0} \ldots {\lambda_k}^{\alpha_k}.$$

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Appear naturally in the expansion

$$1 = 1^d = (\lambda_0 + \cdots + \lambda_k)^d = \sum_{|\alpha|=d} \binom{d}{\alpha} \lambda^\alpha = \sum_{|\alpha|=d} B_\alpha^d.$$

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$\hookrightarrow$ Bernstein coefficients :

$$f = \sum_{|\alpha|=d} b_\alpha(f, d, V) B_\alpha^d.$$

- $b(f, d, V)$: list of coefficients $b_\alpha = b_\alpha(f, d, V)$

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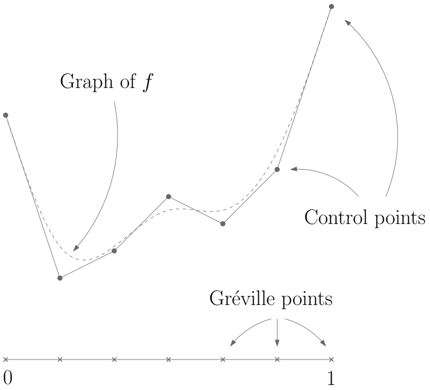

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$\hookrightarrow$ bound on the gap between $f\left(N_\alpha\right)$ and b_α

Gap control net / discrete graph of f

Theorem (08')

The gap between the control net and the discrete graph of f is bounded by

$$\frac{dk(k+2)}{24} \underbrace{\left\|\Delta^2 b(f,d,V)\right\|_\infty}.$$

$$\|$$

$$\max_{\substack{|\gamma| = d-2 \\ 0 \le i < j \le k}} \Big| \underbrace{b_{\gamma+e_i+e_{j-1}} + b_{\gamma+e_{i-1}+e_j} - b_{\gamma+e_i+e_j} - b_{\gamma+e_{i-1}+e_{j-1}}}_{\text{second differences}} \Big|$$

The bound is sharp (attained by a quadratic form).

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Certificate of positivity in the Bernstein basis

If $b(f, d, \Delta) > 0$, then $f > 0$ on Δ.

- Warning : The converse is false in general !
  - $f = 6x^2 - 6x + 2 > 0$ on $[0, 1]$, but $b(f, 2, [0, 1]) = [2, -1, 2]$.

2 Certificates of positivity

- By degree elevation
- By subdivision

Certificates of positivity
by degree elevation

Idea : Express f in the Bernstein basis of degree $D \geq d$, with D getting bigger and bigger.

If D is big enough, then $b(f, d, \Delta) > 0$.

Theorem ('08)

$$D > \frac{d(d-1)k(k+2)}{24m} \left\| \Delta^2 b(f, d, \Delta) \right\|_\infty \Rightarrow b(f, D, \Delta) > 0.$$

2 Certificates of positivity

- By degree elevation
- By subdivision

Certificates of positivity
by subdivision

Idea : Keep the degree constant, and subdivide the simplex Δ.

Tool : successive standard triangulations of degree 2.

If the subdivision is refined enough, then on each subsimplex V^i, $b(f, d, V^i) > 0$.

Theorem ('08)

If $2^N > \frac{k(k+2)}{24\sqrt{m}} \sqrt{dk(k+1)(k+3) \left\|\Delta^2 b(f, d, \Delta)\right\|_\infty}$,

then, after N steps of subdivision, $b(f, D, V^i) > 0$ on each V^i.

Certificates of positivity
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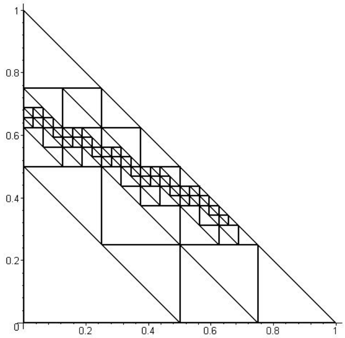

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- better interpolation (ex : $25x^2 + 16y^2 - 40xy - 30x + 24y + 10$)

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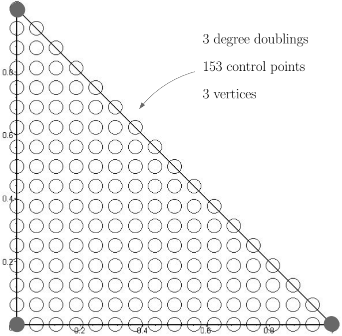

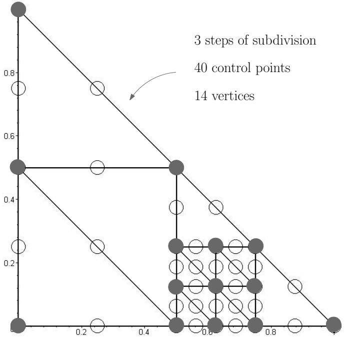

Certificates of positivity
by subdivision

The process stops :

Theorem ('08)

There exists an explicit (computable) $m_{k,d,\tau} > 0$ such that if f has degree $\leq d$ and the bitsize of its coefficients is bounded by τ, then

$$f > m_{k,d,\tau} \text{ on } \Delta.$$

Enclosing property

Let m denote the minimum of f over the standard simplex Δ.

Goal : Enclose m with an arbitrary precision.

Enclosing property

If V is a simplex, and m_V the minimum of f over V, then :

$$m_V \in [s_V, t_V],$$

$$\text{where} \begin{cases} s_V = \min b_\alpha = b_\beta \text{ for some } \beta \\ t_V = \min[f(N_\beta), \underbrace{b_{de_i}}_{=f(V_i)}, i = 0, \ldots, k] \end{cases}$$

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where ε is the aimed precision.

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where ε is the aimed precision.

Tool : Successive standard triangulations of degree 2.

Algorithm

We have a bound on the complexity :

Theorem ('08)

If $2^N > \frac{k(k+2)}{24\sqrt{\varepsilon}} \sqrt{dk(k+1)(k+3)\left\|\Delta^2 b(f,d,\Delta)\right\|_\infty}$,

then at most N steps of subdivision are needed.

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- implemented (in Maple, Maxima)

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- better complexity (as in the univariate case and the multivariate box case)

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Future work

- better complexity (as in the univariate case and the multivariate box case)
- Mathemagix !

Coffee break !

Thank you !